

THE GLOBAL DIMENSION OF THE FULL TRANSFORMATION MONOID WITH AN APPENDIX BY V. MAZORCHUK AND B. STEINBERG

BENJAMIN STEINBERG

ABSTRACT. The representation theory of the symmetric group has been intensively studied for over 100 years and is one of the gems of modern mathematics. The full transformation monoid \mathfrak{T}_n (the monoid of all self-maps of an n -element set) is the monoid analogue of the symmetric group. The investigation of its representation theory was begun by Hewitt and Zuckerman in 1957. Its character table was computed by Putcha in 1996 and its representation type was determined in a series of papers by Ponizovskiĭ, Putcha and Ringel between 1987 and 2000. From their work, one can deduce that the global dimension of $\mathbb{C}\mathfrak{T}_n$ is $n - 1$ for $n = 1, 2, 3, 4$. We prove in this paper that the global dimension is $n - 1$ for all $n \geq 1$ and, moreover, we provide an explicit minimal projective resolution of the trivial module of length $n - 1$.

In an appendix with V. Mazorchuk we compute the indecomposable tilting modules of $\mathbb{C}\mathfrak{T}_n$ with respect to Putcha's quasi-hereditary structure and the Ringel dual (up to Morita equivalence).

1. INTRODUCTION

The character theory of the symmetric group (cf. [JK81, FH91, Mac95, Sag01, CSST10]) is an elegant piece of mathematics, featuring a beautiful blend of algebra and combinatorics, with applications to such diverse areas as probability [Dia88, Dia89] and mathematical physics.

The analogue in monoid theory of the symmetric group is the full transformation monoid \mathfrak{T}_n . This is the monoid of all self-maps of an n -element set. In 1957, Hewitt and Zuckerman initiated the study of the representation theory of \mathfrak{T}_n , showing that the simple $\mathbb{C}\mathfrak{T}_n$ -modules are parameterized by partitions of r where $1 \leq r \leq n$ [HZ57]. However, very few of their results were specific to \mathfrak{T}_n , as witnessed by the fact their main theorem is a special

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case of a result for arbitrary finite monoids obtained at approximately the same time by Munn and Ponizovskii [Pon56, Mun57].

In a *tour de force* work [Put96], Putcha computed the character table of \mathfrak{T}_n and gave an explicit description of all the simple $\mathbb{C}\mathfrak{T}_n$ -modules except for one family. It was the discovery of an explicit description of this second family via exterior powers that led to this paper. Putcha, in fact, knows this explicit description (private communication), but only became aware of it after [Put96, Put98] were written.

Recall that a finite dimensional algebra A has finite representation type if there are only finitely many isomorphism classes of finite dimensional indecomposable A -modules. Ponizovskii proved that $\mathbb{C}\mathfrak{T}_n$ has finite representation type for $n \leq 3$ and conjectured that this was true for all n [Pon87]. Putcha disproved Ponizovskii's conjecture by showing that $\mathbb{C}\mathfrak{T}_n$ does not have finite representation type for $n \geq 5$ [Put98]. He did this by computing enough of the quiver of $\mathbb{C}\mathfrak{T}_n$ to see that $\mathbb{C}\mathfrak{T}_n/\text{rad}^2(\mathbb{C}\mathfrak{T}_n)$ already does not have finite representation type. Putcha also computed the quiver of $\mathbb{C}\mathfrak{T}_4$ and observed that $\mathbb{C}\mathfrak{T}_4/\text{rad}^2(\mathbb{C}\mathfrak{T}_4)$ does have finite representation type. Ringel [Rin00] computed a quiver presentation for $\mathbb{C}\mathfrak{T}_4$ and proved that it is of finite representation type. It is an open question to compute the quiver of $\mathbb{C}\mathfrak{T}_n$ in full generality.

Ringel also proved that $\mathbb{C}\mathfrak{T}_4$ has global dimension 3 by exhibiting the minimal projective resolutions of the simple modules. It is easy to check that the global dimension of $\mathbb{C}\mathfrak{T}_n$ is also $n - 1$ for $n = 1, 2, 3$ using the results of [Put98]. The main result of this paper is that the global dimension of $\mathbb{C}\mathfrak{T}_n$ is $n - 1$ for all $n \geq 1$. As a byproduct of the proof, we also establish that the quiver of $\mathbb{C}\mathfrak{T}_n$ is acyclic. In fact, these results hold *mutatis mutandis* over any ground field of characteristic 0 because \mathbb{Q} is a splitting field for all symmetric groups and hence for all full transformation monoids. In an appendix with Volodymyr Mazorchuk we describe the indecomposable tilting modules of $\mathbb{C}\mathfrak{T}_n$ with respect to Putcha's quasi-hereditary structure [Put98]. They turn out to be precisely the injective indecomposable modules, except the trivial module, and the simple projective one-dimensional module that restricts to the sign representation on permutations and is annihilated by all non-permutations. We also compute the Ringel dual of $\mathbb{C}T_n$.

We prove the main theorem using homological techniques for working with the algebra of a von Neumann regular monoid developed by the author and Margolis in [MS11]. One could alternatively use the theory of quasi-hereditary algebras [CPS88, DR89]. The key point is that outside of a single family the standard modules with respect to the natural quasi-hereditary structure found by Putcha [Put98] are simple and standard modules have good homological properties.

The paper begins with a review of monoid representation theory and the character theory of the full transformation monoid. The following section consists of two parts: the first part proves that the exterior powers of the

natural $\mathbb{C}\mathfrak{T}_n$ -module \mathbb{C}^n are projective indecomposable modules with simple tops and simple radicals that are exterior powers of the augmentation submodule; the second part establishes a vanishing result for higher Ext-functors between simple modules and proves the main result. The appendix, with V. Mazorchuk, uses the results of the previous sections to compute the indecomposable tilting modules and Ringel dual of $\mathbb{C}\mathfrak{T}_n$ with respect to its natural quasi-hereditary structure [Put98].

The reader is referred to [CP61, KRT68, Eil76, Hig92, RS09] for basic semigroup theory. We use [ARS97, Ben98, ASS06] as our primary references for the theory of finite dimensional algebras. In this paper, all modules are assumed to be left modules and all monoids are assumed to act on the left of sets unless otherwise indicated. For a detailed discussion of the representation theory of finite monoids and its applications, the reader is referred to the author's forthcoming book [Ste].

2. THE REPRESENTATION THEORY OF FINITE MONOIDS

This section reviews the necessary background results for the paper. Details can be found in [GMS09, Ste].

2.1. Finite monoids. Fix a finite monoid M . An *ideal* I of M is a non-empty subset I such that $MIM \subseteq I$. Left and right ideals are defined analogously. If $m \in M$, then Mm , mM and MmM are the *principal* left, right and two-sided ideals generated by m , respectively. We put

$$I(m) = \{n \in M \mid m \notin MnM\}.$$

If $I(m) \neq \emptyset$, then it is an ideal.

The following equivalence relations are three of *Green's relations* [Gre51]. Set, for $m_1, m_2 \in M$,

- (i) $m_1 \mathcal{J} m_2$ if and only if $Mm_1M = Mm_2M$;
- (ii) $m_1 \mathcal{L} m_2$ if and only if $Mm_1 = Mm_2$;
- (iii) $m_1 \mathcal{R} m_2$ if and only if $m_1M = m_2M$.

The \mathcal{J} -class of an element m is denoted by J_m , and similarly the \mathcal{L} -class and \mathcal{R} -class of m are denoted L_m and R_m , respectively.

Let us describe Green's relations on \mathfrak{T}_n ; this is a standard exercise in semigroup theory (cf. [CP61, GM09]). The *rank* of a mapping $f \in \mathfrak{T}_n$ is the cardinality of its image. Let $f, g \in \mathfrak{T}_n$. Then $f \mathcal{J} g$ if and only if they have the same rank, $f \mathcal{R} g$ if and only if they have the same image and $f \mathcal{L} g$ if and only if they induce the same partition of the domain into fibers.

The set of idempotents of a monoid M is denoted $E(M)$. If $e \in E(M)$, then eMe is a monoid with identity e . Let G_e be the group of units of eMe . It is called the *maximal subgroup* of M at e . If $e, f \in E(M)$ with $MeM = MfM$, then $eMe \cong fMf$ and $G_e \cong G_f$ (cf. [Til76]).

The group of units of \mathfrak{T}_n is the symmetric group \mathfrak{S}_n . If $e \in \mathfrak{T}_n$ has rank r , then it is well known and easy to show that $e\mathfrak{T}_ne \cong \mathfrak{T}_r$ and hence $G_e \cong \mathfrak{S}_r$.

A standard fact in the theory of finite monoids is the following [Til76].

Proposition 2.1. *Let $e \in E(M)$. Then $J_e \cap eMe = G_e$, $J_e \cap Me = L_e$ and $J_e \cap eM = R_e$.*

The maximal subgroup G_e at an idempotent e acts freely on the right of L_e and on the left of R_e , respectively, by multiplication and two elements of one of these sets are in the same G_e -orbit if and only if they are \mathcal{R} -equivalent, respectively, \mathcal{L} -equivalent; see [CP61] or [RS09, Appendix A].

A monoid M is (von Neuman) *regular* if, for all $m \in M$, there exists $m' \in M$ with $mm'm = m$. For example, it is well known that the full transformation monoid \mathfrak{T}_n is regular [CP61, GM09], as is the monoid $M_n(\mathbb{F})$ of all $n \times n$ matrices over a field \mathbb{F} [CP61]. A finite monoid is regular if and only if each \mathcal{J} -class contains an idempotent, cf. [KRT68] or [RS09, Appendix A].

2.2. Simple modules. Let \mathbb{k} be a field. We continue to hold fixed a finite monoid M . If $X \subseteq M$ we let $\mathbb{k}X$ denote the \mathbb{k} -linear span of X in the monoid algebra $\mathbb{k}M$. If $I \subseteq M$ is a left (respectively, right or two-sided) ideal of M , then $\mathbb{k}I$ is a left (respectively, right or two-sided) ideal of $\mathbb{k}M$.

Let S be a simple $\mathbb{k}M$ -module. We say that an idempotent $e \in E(M)$ is an *apex* for S if $eS \neq 0$ and $\mathbb{k}I(e)S = 0$. One has that $mS = 0$ if and only if $m \in I(e)$. It follows that if e, f are apexes of S , then $MeM = MfM$.

Fix an idempotent $e \in E(M)$ and put $A_e = \mathbb{k}M/\mathbb{k}I(e)$. Observe that $eA_ee \cong \mathbb{k}[eMe]/\mathbb{k}[eI(e)e] \cong \mathbb{k}G_e$ by Proposition 2.1. A simple $\mathbb{k}M$ -module S with apex e is the same thing as a simple A_e -module S with $eS \neq 0$. One can then apply the theory [Gre80, Chapter 6] to classify these modules. This was done in [GMS09]. See [Ste, Chapters 4,5] for details.

Notice that as \mathbb{k} -vector spaces we have that $A_e e \cong \mathbb{k}L_e$ and $eA_e \cong \mathbb{k}R_e$ [GMS09]. The corresponding left $\mathbb{k}M$ -module structure on $\mathbb{k}L_e$ is defined by

$$m \odot \ell = \begin{cases} m\ell, & \text{if } m\ell \in L_e \\ 0, & \text{else} \end{cases}$$

for $m \in M$ and $\ell \in L_e$. From now on we will omit the symbol “ \odot .” The right $\mathbb{k}M$ -module structure on $\mathbb{k}R_e$ is defined dually. In the semigroup theory literature, $\mathbb{k}L_e$ and $\mathbb{k}R_e$ are known as left and right *Schützenberger representations*. Note that $\mathbb{k}L_e$ is a free right $\mathbb{k}G_e$ -module and $\mathbb{k}R_e$ is a free left $\mathbb{k}G_e$ -module because G_e acts freely on the right of L_e and on the left of R_e . In fact, $\mathbb{k}L_e$ is a $\mathbb{k}M$ - $\mathbb{k}G_e$ -bimodule and dually $\mathbb{k}R_e$ is a $\mathbb{k}G_e$ - $\mathbb{k}M$ -bimodule. Thus we can define functors

$$\begin{aligned} \text{Ind}_{G_e} &: \mathbb{k}G_e\text{-mod} \longrightarrow \mathbb{k}M\text{-mod} \\ \text{Coind}_{G_e} &: \mathbb{k}G_e\text{-mod} \longrightarrow \mathbb{k}M\text{-mod} \\ \text{Res}_{G_e} &: \mathbb{k}M\text{-mod} \longrightarrow \mathbb{k}G_e\text{-mod} \\ T_e &: \mathbb{k}M\text{-mod} \longrightarrow \mathbb{k}M\text{-mod} \\ N_e &: \mathbb{k}M\text{-mod} \longrightarrow \mathbb{k}M\text{-mod} \end{aligned}$$

by putting

$$\begin{aligned} \text{Ind}_{G_e}(V) &= A_e e \otimes_{eA_e e} V = \mathbb{k}L_e \otimes_{\mathbb{k}G_e} V \\ \text{Coind}_{G_e}(V) &= \text{Hom}_{eA_e e}(eA_e, V) = \text{Hom}_{\mathbb{k}G_e}(\mathbb{k}R_e, V) \\ \text{Res}_{G_e}(V) &= eV = \text{Hom}_{A_e}(A_e e, V) = eA_e \otimes_{A_e} V \\ T_e(V) &= \mathbb{k}MeV \\ N_e(V) &= \{v \in V \mid e\mathbb{k}Mv = 0\}. \end{aligned}$$

One has that $\text{Res}_{G_e}(\text{Ind}_{G_e}(V)) \cong V \cong \text{Res}_{G_e}(\text{Coind}_{G_e}(V))$ for any $\mathbb{k}G_e$ -module, cf. [GMS09]. Note that the functors Ind_{G_e} and Coind_{G_e} are exact (cf. [GMS09] or [Ste, Chapters 4,5]) because $\mathbb{k}L_e$ and $\mathbb{k}R_e$ are free $\mathbb{k}G_e$ -modules (on the appropriate sides). They also preserve indecomposability (see [Ste, Chapters 4,5]).

If V is a module over a finite dimensional algebra A , then $\text{rad}(V)$ denotes the radical of V and $\text{soc}(V)$ the socle of V (see [ASS06] for the definitions).

We now state the fundamental theorem of Clifford-Munn-Ponizovskii theory [CP61, Chapter 5], as formulated in [GMS09]; see also [Ste, Chapter 5].

Theorem 2.2. *Let M be a finite monoid and \mathbb{k} a field.*

- (i) *There is a bijection between isomorphism classes of simple $\mathbb{k}M$ -modules with apex $e \in E(M)$ and isomorphism classes of simple $\mathbb{k}G_e$ -modules given on a simple $\mathbb{k}M$ -module S with apex e and a simple $\mathbb{k}G_e$ -module V by*

$$S \longmapsto \text{Res}_{G_e}(S) = eS$$

$$\begin{aligned} V \longmapsto V^\# &= \text{Ind}_{G_e}(V)/N_e(\text{Ind}_{G_e}(V)) = \text{Ind}_{G_e}(V)/\text{rad}(\text{Ind}_{G_e}(V)) \\ &\cong \text{soc}(\text{Coind}_{G_e}(V)) = T_e(\text{Coind}_{G_e}(V)). \end{aligned}$$

- (ii) *Every simple $\mathbb{k}M$ -module has an apex (unique up to \mathcal{J} -equivalence).*
- (iii) *If V is a simple $\mathbb{k}G_e$ -module, then every composition factor of $\text{Ind}_{G_e}(V)$ and $\text{Coind}_{G_e}(V)$ has apex f with $MeM \subseteq MfM$. Moreover, $V^\#$ is the unique composition factor of either of these two $\mathbb{k}M$ -modules with apex e and it appears in both these modules as a composition factor with multiplicity one.*

If we denote the set of isomorphism classes of simple $\mathbb{k}M$ -modules by $\text{Irr}_{\mathbb{k}}(M)$, then there is the following parametrization of the irreducible representations of M .

Corollary 2.3. *Let e_1, \dots, e_s be a complete set of idempotent representatives of the \mathcal{J} -classes of M containing idempotents. Then there is a bijection between $\text{Irr}_{\mathbb{k}}(M)$ and the disjoint union $\bigcup_{i=1}^s \text{Irr}_{\mathbb{k}}(G_{e_i})$.*

If M is regular and \mathbb{k} has characteristic 0, then the modules $\text{Ind}_{G_e}(V)$ and $\text{Coind}_{G_e}(V)$ with $e \in E(M)$ and $V \in \text{Irr}_{\mathbb{k}}(G_e)$ form the standard and costandard modules, respectively, for the natural structure quasi-hereditary algebra on $\mathbb{k}M$ found by Putcha [Put98]. See [CPS88, DR89] for more on quasi-hereditary algebras.

2.3. Homological aspects. Let A be a finite dimensional \mathbb{k} -algebra. The *projective dimension* $\text{pd } V$ of an A -module V is the minimum length (possibly infinite) of a projective resolution of V . Each finite dimensional A -module V has a unique minimal projective resolution (minimal in both length and in a certain categorical sense); see [ARS97, Ben98, ASS06]. Formally, a projective resolution $P_\bullet \rightarrow V$ is *minimal* if each boundary map $d_n: P_n \rightarrow d_n(P_n)$ is a projective cover. The following well-known proposition is stated in the context of group algebras in [CTVEZ03, Proposition 3.2.3], but the proof there is valid for any finite dimensional algebra.

Proposition 2.4. *Let A be a finite dimensional \mathbb{k} -algebra, M a finite dimensional A -module and let $P_\bullet \rightarrow M$ be a projective resolution. Then the following are equivalent.*

- (1) $P_\bullet \rightarrow M$ is the minimal projective resolution of M .
- (2) $\text{Hom}_A(P_q, S) \cong \text{Ext}_A^q(M, S)$ for any $q \geq 0$ and simple A -module S .

The *global dimension* of A is

$$\text{gl. dim } A = \sup\{\text{pd } V \mid V \text{ is an } A\text{-module}\}.$$

It is convenient to reformulate the definition in terms of the Ext-functors. The Ext-functors measure the failure of the Hom-functors to be exact. Details about their properties can be found in [Ben98, ASS06] or any text on homological algebra. A classical fact is that

$$\text{gl. dim } A = \sup\{n \in \mathbb{N} \mid \text{Ext}_A^n(S, S') \neq 0, \text{ for some simple modules } S, S'\}$$

where the supremum could be infinite [ARS97, Ben98, ASS06]. This reformulation relies on the following well-known lemma, which is proved by induction on the number of composition factors using the long exact sequence associated to the Ext-functors.

Lemma 2.5. *Let V, W be finite dimensional A -modules. Then one has that $\text{Ext}_A^n(V, W) = 0$ if either of the following two conditions hold.*

- (i) $\text{Ext}_A^n(V, S) = 0$ for each composition factor S of W .
- (ii) $\text{Ext}_A^n(S', W) = 0$ for each composition factor S' of V .

The following result is [MS11, Lemma 3.3].

Lemma 2.6. *Let M be a finite regular monoid and \mathbb{k} a field. Let I be an ideal of M . Then the isomorphism*

$$\text{Ext}_{\mathbb{k}M}^n(V, W) \cong \text{Ext}_{\mathbb{k}M/\mathbb{k}I}^n(V, W)$$

holds for any $\mathbb{k}M/\mathbb{k}I$ -modules V, W and all $n \geq 0$.

The author and Margolis proved in [MS11, Lemma 3.5] the following lemma in the same vein as the Eckmann-Shapiro lemma from group cohomology.

Lemma 2.7. *Let M be a finite regular monoid and \mathbb{k} a field. Let $e \in E(M)$ and $I = MeM \setminus J_e$. Then, for any $\mathbb{k}G_e$ -module V and $\mathbb{k}M/\mathbb{k}I$ -module W , one has natural isomorphisms*

$$\begin{aligned} \text{Ext}_{\mathbb{k}M}^n(\text{Ind}_{G_e}(V), W) &\cong \text{Ext}_{\mathbb{k}G_e}^n(V, \text{Res}_{G_e}(W)) \\ \text{Ext}_{\mathbb{k}M}^n(W, \text{Coind}_{G_e}(V)) &\cong \text{Ext}_{\mathbb{k}G_e}^n(\text{Res}_{G_e}(W), V) \end{aligned}$$

for all $n \geq 0$.

Since $\mathbb{k}G_e$ is semisimple whenever \mathbb{k} is of characteristic zero, we obtain the following corollary.

Corollary 2.8. *Let M be a finite regular monoid and \mathbb{k} a field of characteristic 0. Let $e \in E(M)$ and let $I = MeM \setminus J_e$. Then, for any $\mathbb{k}G_e$ -module V and $\mathbb{k}M/\mathbb{k}I$ -module W , one has*

$$\begin{aligned} \text{Ext}_{\mathbb{k}M}^n(\text{Ind}_{G_e}(V), W) &= 0 \\ \text{Ext}_{\mathbb{k}M}^n(W, \text{Coind}_{G_e}(V)) &= 0 \end{aligned}$$

for all $n \geq 1$.

Nico [Nic71, Nic72] proved that the global dimension of a regular monoid over a field of characteristic zero is always finite.

Theorem 2.9 (Nico). *Let M be a finite regular monoid and let \mathbb{k} be a field of characteristic 0. Then $\text{gl. dim } \mathbb{k}M$ is bounded by $2(m-1)$ where m is the length of the longest chain of non-zero principal ideals of M .*

2.4. The character theory of the full transformation monoid. The character theory of \mathfrak{T}_n has a very long history, beginning with the work of Hewitt and Zuckerman [HZ57]. A complete computation of the character table of \mathfrak{T}_n was finally achieved by Putcha nearly 40 years later [Put96, Theorem 2.1]. To formulate his result, first let $e_r \in \mathfrak{T}_n$ be the idempotent given by

$$e_r(i) = \begin{cases} i, & \text{if } i \leq r \\ 1, & \text{if } i > r \end{cases} \quad (2.1)$$

and note that e_1, \dots, e_n form a complete set of idempotent representatives of the \mathcal{J} -classes of \mathfrak{T}_n and $e_r \mathfrak{T}_n e_r \cong \mathfrak{T}_r$, whence $G_{e_r} \cong \mathfrak{S}_r$. The isomorphism takes $f \in e_r \mathfrak{T}_n e_r$ to $f|_{[r]}$ where $[r] = \{1, \dots, r\}$.

The reader is referred to [JK81] for the representation theory of the symmetric group. If λ is a partition of n , then $S(\lambda)$ will denote the corresponding simple module (Specht module). We put $L(\lambda) = S(\lambda)^\sharp$ to avoid cumbersome notation. Let us use 1^k as short hand for a sequence of k ones occurring in a partition. With this notation, $S((1^r))$ is the sign representation of \mathfrak{S}_r . On the other hand, $S((r))$ is the trivial representation of \mathfrak{S}_r . Thus we consider $S((1))$ to be both the trivial and the sign representation of \mathfrak{S}_1 .

Theorem 2.10 (Putcha). *Fix $n \geq 1$ and let $S(\lambda) \in \text{Irr}_{\mathbb{C}}(\mathfrak{S}_r)$ for $1 \leq r \leq n$.*

- (i) If $\lambda \neq (1^r)$, then $\text{Ind}_{\mathfrak{S}_r}(S(\lambda))$ is simple (and hence equal to $L(\lambda)$). Moreover, its restriction to $\mathbb{C}\mathfrak{S}_n$ is isomorphic to the induced module $\mathbb{C}\mathfrak{S}_n \otimes_{\mathbb{C}[\mathfrak{S}_r \times \mathfrak{S}_{n-r}]} (S(\lambda) \otimes S((n-r)))$.
- (ii) If $\lambda = (1^r)$, then the restriction of $L((1^r))$ to $\mathbb{C}\mathfrak{S}_n$ is the simple module $S((n-r+1, 1^{r-1}))$ of dimension $\binom{n-1}{r-1}$.

Let us remark that Putcha [Put96] uses very different notation than ours. If θ is an irreducible representation of \mathfrak{S}_r afforded by a simple module V , then Putcha uses θ^+ to denote the representation afforded by $\text{Ind}_{\mathfrak{S}_r}(V)$ and θ^- to denote the representation afforded by $\text{Coind}_{\mathfrak{S}_r}(V)$. He uses $\tilde{\theta}$ for the representation afforded by V^\sharp . A proof of Theorem 2.10 can be found in [Ste, Chapter 5].

We proceed to give a new proof of Theorem 2.10(ii), which is simpler than Putcha's, by exhibiting the simple module. Putcha has informed the author that he is aware of this construction. The key observation is that $S((n-r+1, 1^{r-1}))$ is an exterior power of the representation $S((n-1, 1))$ of \mathfrak{S}_n .

Note that \mathbb{C}^n is a $\mathbb{C}\mathfrak{T}_n$ -module by defining $fv_i = v_{f(i)}$, for $f \in \mathfrak{T}_n$, where v_1, \dots, v_n denotes the standard basis for \mathbb{C}^n . We call this the *natural module*. Moreover, the *augmentation*

$$\text{Aug}(\mathbb{C}^n) = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_1 + \dots + x_n = 0\}$$

is a $\mathbb{C}\mathfrak{T}_n$ -submodule and $\mathbb{C}^n / \text{Aug}(\mathbb{C}^n)$ is the trivial $\mathbb{C}\mathfrak{T}_n$ -module.

Theorem 2.11. *Let $n \geq 1$ and let $V = \text{Aug}(\mathbb{C}^n)$. Then the exterior power $\Lambda^{r-1}(V)$, for $1 \leq r \leq n$, is a simple $\mathbb{C}\mathfrak{T}_n$ -module with apex e_r and with $e_r \Lambda^{r-1}(V) \cong S((1^r))$ the sign representation of $G_{e_r} \cong \mathfrak{S}_r$. In other words, $L((1^r)) = \Lambda^{r-1}(V)$.*

Proof. If $1 \leq m \leq n$, then clearly $e_m \mathbb{C}^n \cong \mathbb{C}^m$ and $e_m \text{Aug}(\mathbb{C}^n) \cong \text{Aug}(\mathbb{C}^m)$ under the identification of $e_m \mathbb{C}\mathfrak{T}_n e_m = \mathbb{C}[e_m \mathfrak{T}_n e_m]$ with $\mathbb{C}\mathfrak{T}_m$ induced by restricting an element of $e_m \mathfrak{T}_n e_m$ to $[m]$. In particular, as $e_m \Lambda^{r-1}(V) = \Lambda^{r-1}(e_m V)$ and $\dim e_m V = \dim \text{Aug}(\mathbb{C}^m) = m - 1$, we conclude that $e_m \Lambda^{r-1}(V) = 0$ if $m < r$.

Let us now assume that $m \geq r$. By [FH91, Proposition 3.12], the exterior power $\Lambda^{r-1}(e_m V) \cong \Lambda^{r-1}(\text{Aug}(\mathbb{C}^m))$ is a simple $\mathbb{C}\mathfrak{S}_m$ -module of degree $\binom{m-1}{r-1}$ and by [FH91, Exercise 4.6] it is, in fact, the Specht module $S((m-r+1, 1^{r-1}))$. In particular, we have that $e_r \Lambda^{r-1}(V) = S((1^r))$. It follows that $W = \Lambda^{r-1}(V)$ is a simple $\mathbb{C}\mathfrak{T}_n$ -module with apex e_r and that $W \cong L((1^r))$. \square

3. THE GLOBAL DIMENSION OF $\mathbb{C}\mathfrak{T}_n$

In this section, we prove that the global dimension of $\mathbb{C}\mathfrak{T}_n$ is $n - 1$. Our first goal is to provide the minimal projective resolutions of the exterior powers of $\text{Aug}(\mathbb{C}^n)$. We retain the notation of the previous section.

3.1. Minimal projective resolutions of the exterior powers. We continue to denote the natural $\mathbb{C}\mathfrak{T}_n$ -module by \mathbb{C}^n . It turns out that each exterior power of \mathbb{C}^n is a projective indecomposable module.

Theorem 3.1. *For each r with $1 \leq r \leq n$, the $\mathbb{C}\mathfrak{T}_n$ -module $\Lambda^r(\mathbb{C}^n)$ is a projective indecomposable module.*

Proof. Let e_r be the idempotent from (2.1) and $W = \mathbb{C}^n$. Let

$$\eta = \frac{1}{r!} \sum_{g \in G_{e_r}} \text{sgn}(g|_{[r]})g.$$

We will prove that η is a primitive idempotent of $\mathbb{C}\mathfrak{T}_n$ and that $\mathbb{C}\mathfrak{T}_n\eta \cong \Lambda^r(W)$.

Again denote by v_1, \dots, v_n the standard basis for W . Observe that $\mathfrak{T}_n e_r$ consists of all the mappings $f \in \mathfrak{T}_n$ with $f(x) = f(1)$ for $x > r$. We can thus define an isomorphism of $\mathbb{C}\mathfrak{T}_n$ -modules $\rho: \mathbb{C}\mathfrak{T}_n e_r \rightarrow W^{\otimes r}$ by

$$\rho(f) = v_{f(1)} \otimes \cdots \otimes v_{f(r)}$$

for $f \in \mathfrak{T}_n e_r$. We can identify G_{e_r} with the symmetric group \mathfrak{S}_r via the isomorphism $g \mapsto g|_{[r]}$. Under this identification, we see that if $g \in G_{e_r}$ and $f \in \mathfrak{T}_n e_r$, then

$$\rho(fg) = v_{f(g(1))} \otimes \cdots \otimes v_{f(g(r))} = \rho(f)g|_{[r]}$$

where \mathfrak{S}_r acts on the right of $W^{\otimes r}$ by permuting the tensor factors.

We have that η is a primitive idempotent of $\mathbb{C}G_{e_r}$ and $\mathbb{C}G_{e_r}\eta$ affords the sign representation of $G_{e_r} \cong \mathfrak{S}_r$. By definition, $\Lambda^r(W) = W^{\otimes r} \otimes_{\mathbb{C}\mathfrak{S}_r} S((1^r))$ and hence

$$\Lambda^r(\mathbb{C}^n) \cong \mathbb{C}\mathfrak{T}_n e_r \otimes_{\mathbb{C}G_{e_r}} S((1^r)) = \mathbb{C}\mathfrak{T}_n e_r \otimes_{\mathbb{C}G_{e_r}} \mathbb{C}G_{e_r}\eta \cong \mathbb{C}\mathfrak{T}_n\eta.$$

Since $\eta \in \mathbb{C}\mathfrak{T}_n$ is an idempotent, we deduce that $\Lambda^r(\mathbb{C}^n)$ is a projective $\mathbb{C}\mathfrak{T}_n$ -module.

It remains to prove that η is primitive or, equivalently, that $\eta\mathbb{C}\mathfrak{T}_n\eta$ is a local ring. In fact, we show that it is isomorphic to \mathbb{C} . Indeed, as a $\mathbb{C}G_{e_r}$ -module we have that

$$e_r\mathbb{C}\mathfrak{T}_n\eta \cong e_r\Lambda^r(W) = \Lambda^r(e_r W) \cong S((1^r))$$

and hence there is a vector space isomorphism $\eta\mathbb{C}\mathfrak{T}_n\eta \cong \eta S((1^r)) \cong \mathbb{C}$, as η is the primitive idempotent of $\mathbb{C}G_{e_r}$ corresponding to $S((1^r))$. Since $\dim \eta\mathbb{C}\mathfrak{T}_n\eta = 1$, we conclude that $\eta\mathbb{C}\mathfrak{T}_n\eta \cong \mathbb{C}$ as a \mathbb{C} -algebra. This completes the proof that η is primitive and hence $\Lambda^r(W)$ is a projective indecomposable $\mathbb{C}\mathfrak{T}_n$ -module. \square

A crucial observation for understanding the representation theory of \mathfrak{T}_n is the following short exact sequence for the projective indecomposable module $\Lambda^r(\mathbb{C}^n)$, which can be viewed as a categorification of Pascal's identity for binomial coefficients.

Theorem 3.2. *Let $1 \leq r \leq n$ and let $V = \text{Aug}(\mathbb{C}^n)$. Then there is a short exact sequence of $\mathbb{C}\mathfrak{S}_n$ -modules*

$$0 \longrightarrow \Lambda^r(V) \longrightarrow \Lambda^r(\mathbb{C}^n) \longrightarrow \Lambda^{r-1}(V) \longrightarrow 0$$

which does not split if $1 \leq r < n$. Note that $\Lambda^n(\mathbb{C}^n) \cong \Lambda^{n-1}(V)$ is the projective simple module $L((1^n))$.

Proof. Clearly, $\Lambda^r(V)$ is a submodule of $\Lambda^r(\mathbb{C}^n)$. Let v_1, \dots, v_n be the standard basis for \mathbb{C}^n and put $w_i = v_i - v_n$ for $i = 1, \dots, n-1$. Then w_1, \dots, w_{n-1} is a basis for V . We claim that $\Lambda^r(\mathbb{C}^n)/\Lambda^r(V) \cong \Lambda^{r-1}(V)$.

Put $w_n = v_1 + \dots + v_n$. Then w_1, \dots, w_n is a basis for \mathbb{C}^n . Define $\rho: \Lambda^r(\mathbb{C}^n) \longrightarrow \Lambda^{r-1}(V)$ on the basis of r -fold wedge products of w_1, \dots, w_n by

$$\rho(w_{i_1} \wedge \dots \wedge w_{i_r}) = \begin{cases} w_{i_1} \wedge \dots \wedge w_{i_{r-1}}, & \text{if } i_r = n \\ 0, & \text{else} \end{cases}$$

for $1 \leq i_1 < \dots < i_r \leq n$. This is clearly a surjective linear map with kernel $\Lambda^r(V)$. Let us check that it is a $\mathbb{C}\mathfrak{S}_n$ -module homomorphism. If $g \in \mathfrak{S}_n$, then since \mathbb{C}^n/V is the trivial module, it follows that $gw_n + V = w_n + V$ and so $gw_n = w_n + v_g$ with $v_g \in V$. Therefore, if $1 \leq i_1 < \dots < i_{r-1} \leq n-1$, then

$$\begin{aligned} g(w_{i_1} \wedge \dots \wedge w_{i_{r-1}} \wedge w_n) &= gw_{i_1} \wedge \dots \wedge gw_{i_{r-1}} \wedge w_n + \\ &\quad gw_{i_1} \wedge \dots \wedge gw_{i_{r-1}} \wedge v_g \\ &\in gw_{i_1} \wedge \dots \wedge gw_{i_{r-1}} \wedge w_n + \Lambda^r(V). \end{aligned}$$

We conclude that

$$g\rho(w_{i_1} \wedge \dots \wedge w_{i_{r-1}} \wedge w_n) = gw_{i_1} \wedge \dots \wedge gw_{i_{r-1}} = \rho(g(w_{i_1} \wedge \dots \wedge w_{i_{r-1}} \wedge w_n)).$$

This completes the proof that there is such an exact sequence. If $1 \leq r < n$, then it cannot split because $\Lambda^r(\mathbb{C}^n)$ is indecomposable by Theorem 3.1. \square

Let us deduce the following important corollary.

Corollary 3.3. *For $1 \leq r \leq n$, the exterior power $\Lambda^r(\mathbb{C}^n)$ is a projective indecomposable module with $\text{rad}(\Lambda^r(\mathbb{C}^n)) = \Lambda^r(\text{Aug}(\mathbb{C}^n))$ and simple top $\Lambda^{r-1}(\text{Aug}(\mathbb{C}^n))$.*

Proof. Theorem 3.1 yields that $P = \Lambda^r(\mathbb{C}^n)$ is a projective indecomposable module and hence has a simple top. As the modules $\Lambda^r(\text{Aug}(\mathbb{C}^n))$ and $\Lambda^{r-1}(\text{Aug}(\mathbb{C}^n))$ are simple by Theorem 2.11, Theorem 3.2 yields that $\text{rad}(P) = \Lambda^r(\text{Aug}(\mathbb{C}^n))$ and $P/\text{rad}(P) \cong \Lambda^{r-1}(\text{Aug}(\mathbb{C}^n))$. \square

Theorem 3.2 allows us to construct the minimal projective resolution of $\Lambda^{r-1}(\text{Aug}(\mathbb{C}^n))$ for $1 \leq r \leq n$.

Corollary 3.4. *Let v_1, \dots, v_n be the standard basis for \mathbb{C}^n , let $w_i = v_i - v_n$ for $1 \leq i \leq n-1$ and let $w_n = v_1 + \dots + v_n$. Let $V = \text{Aug}(\mathbb{C}^n)$. Then, for*

$1 \leq r \leq n$, the minimal projective resolution of the simple module $\Lambda^{r-1}(V)$ is

$$0 \longrightarrow P_{n-r} \xrightarrow{d_{n-r}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} \Lambda^{r-1}(V) \longrightarrow 0$$

where $P_q = \Lambda^{q+r}(\mathbb{C}^n)$ and

$$d_q(w_{i_1} \wedge \cdots \wedge w_{i_{q+r}}) = \begin{cases} w_{i_1} \wedge \cdots \wedge w_{i_{q+r-1}}, & \text{if } i_{q+r} = n \\ 0, & \text{else.} \end{cases}$$

Therefore, $\text{pd } \Lambda^{r-1}(V) = n-r$ for $1 \leq r \leq n$. In particular, the projective dimension of the trivial $\mathbb{C}\mathfrak{T}_n$ -module is $n-1$.

Proof. The exactness of the resolution follows from repeated application of Theorem 3.2 (and its proof). Corollary 3.3 implies that each mapping

$$d_q: \Lambda^{q+r}(\mathbb{C}^n) \longrightarrow d_q(\Lambda^{q+r}(\mathbb{C}^n)) = \Lambda^{q+r-1}(V)$$

is a projective cover and so the resolution is a minimal projective resolution. The final statement holds because $\Lambda^0(V)$ is the trivial $\mathbb{C}\mathfrak{T}_n$ -module. \square

It follows from Corollary 3.4 that the global dimension of $\mathbb{C}\mathfrak{T}_n$ is at least $n-1$. The next subsection will show that this lower bound is tight. In fact, Corollary 3.4 yields that the cohomological dimension of \mathfrak{T}_n over \mathbb{C} is $n-1$ (the cohomological dimension of a monoid M over a base ring R is the projective dimension of the trivial RM -module).

3.2. A computation of the global dimension. We first compute Ext from an exterior power of the augmentation submodule of \mathbb{C}^n .

Proposition 3.5. *For $1 \leq k, r \leq n$ and λ a partition of k , one has that*

$$\text{Ext}_{\mathbb{C}\mathfrak{T}_n}^m(L((1^r)), L(\lambda)) \cong \begin{cases} \mathbb{C}, & \text{if } \lambda = (1^{r+m}) \\ 0, & \text{else.} \end{cases}$$

Proof. Let $V = \text{Aug}(\mathbb{C}^n)$. Recall that $L((1^r)) \cong \Lambda^{r-1}(V)$ by Theorem 2.11. Using the minimal projective resolution for $\Lambda^{r-1}(V)$ from Corollary 3.4 and Proposition 2.4, we deduce that

$$\begin{aligned} \text{Ext}_{\mathbb{C}\mathfrak{T}_n}^m(\Lambda^{r-1}(V), L(\lambda)) &\cong \text{Hom}_{\mathbb{C}\mathfrak{T}_n}(\Lambda^{r+m}(\mathbb{C}^n), L(\lambda)) \\ &\cong \text{Hom}_{\mathbb{C}\mathfrak{T}_n}(\Lambda^{r+m-1}(V), L(\lambda)) \end{aligned}$$

where the last isomorphism uses that $\Lambda^{r+m}(\mathbb{C}^n)$ has simple top $\Lambda^{r+m-1}(V)$ by Corollary 3.3. Recalling that $\Lambda^{r+m-1}(V) \cong L((1^{r+m}))$ by Theorem 2.11, the result follows. \square

We next prove a vanishing result when the first variable is not one of the exterior powers of the augmentation submodule of \mathbb{C}^n (and hence is an induced module).

Proposition 3.6. *Let $n \geq 1$ and $1 \leq k, r \leq n$. Let $S(\lambda)$ be a simple $\mathbb{C}\mathfrak{S}_r$ -module with $\lambda \neq (1^r)$ and let $S(\mu)$ be a simple $\mathbb{C}\mathfrak{S}_k$ -module. Then $\text{Ext}_{\mathbb{C}\mathfrak{T}_n}^m(L(\lambda), L(\mu)) = 0$ unless $0 \leq m \leq r-k \leq n-1$.*

Proof. Suppose first that $k \geq r$. Let $I_{r-1} \subseteq \mathfrak{T}_n$ be the ideal of mappings of rank at most $r-1$, where we take $I_0 = \emptyset$, and put $A = \mathbb{C}\mathfrak{T}_n/\mathbb{C}I_{r-1}$. Since $L(\lambda) = \text{Ind}_{G_{e_r}}(S(\lambda))$ by Theorem 2.10 and $L(\mu)$ is an A -module, we conclude that $\text{Ext}_{\mathbb{C}\mathfrak{T}_n}^m(L(\lambda), L(\mu)) = 0$ for all $m \geq 1$ by Corollary 2.8. Since there are no homomorphisms from $L(\lambda)$ to $L(\mu)$ if $k > r$, this completes the first case.

Next suppose that $k \leq r$. We proceed by induction on $r-k$ where the base case $r=k$ has already been handled. Assume that $k < r$ and that the result is true for k' with $k < k' \leq n$. Let $A = \mathbb{C}\mathfrak{T}_n/\mathbb{C}I_{k-1}$ and consider the exact sequence of A -modules

$$0 \longrightarrow L(\mu) \longrightarrow \text{Coind}_{G_{e_k}}(S(\mu)) \longrightarrow \text{Coind}_{G_{e_k}}(S(\mu))/L(\mu) \longrightarrow 0.$$

Since $L(\lambda)$ is an A -module, Corollary 2.8 and the long exact sequence for Ext imply that, for $m > r-k \geq 1$,

$$\text{Ext}_{\mathbb{C}\mathfrak{T}_n}^m(L(\lambda), L(\mu)) \cong \text{Ext}_{\mathbb{C}\mathfrak{T}_n}^{m-1}(L(\lambda), \text{Coind}_{G_{e_k}}(S(\mu))/L(\mu)) = 0$$

where the last equality uses that each composition factor of the module $\text{Coind}_{G_{e_k}}(S(\mu))/L(\mu)$ has apex $e_{k'}$ with $k' > k$ by Theorem 2.2, induction and Lemma 2.5. This completes the proof. \square

We are now prepared to prove the main result of the paper.

Theorem 3.7. *The global dimension of $\mathbb{C}\mathfrak{T}_n$ is $n-1$ for all $n \geq 1$.*

Proof. By Proposition 3.5, we have that $\text{Ext}_{\mathbb{C}\mathfrak{T}_n}^{n-1}(L((1)), L((1^n))) \cong \mathbb{C}$. On the other hand, Proposition 3.5 and Proposition 3.6 yield that

$$\text{Ext}_{\mathbb{C}\mathfrak{T}_n}^m(L(\lambda), L(\mu)) = 0$$

for all simple modules $L(\lambda), L(\mu)$ and $m \geq n$. This completes the proof that $\text{gl. dim } \mathbb{C}\mathfrak{T}_n = n-1$. \square

We remark that since $L((1))$ is a simple injective module (because it is isomorphic to $\text{Hom}_{\mathbb{C}}(e_1\mathbb{C}\mathfrak{T}_n, \mathbb{C})$ where e_1 is the constant mapping to 1) it follows that if $\lambda \neq (1)$, then $\text{Ext}_{\mathbb{C}\mathfrak{T}_n}^{n-1}(L(\lambda), L(\mu)) = 0$ for all partitions μ and hence no simple module other than the trivial module has projective dimension $n-1$.

In fact, since \mathbb{Q} is a splitting field for all symmetric groups, and hence for all full transformation monoids, the above argument works *mutatis mutandis* to prove that $\mathbb{k}\mathfrak{T}_n$ has global dimension $n-1$ for all fields \mathbb{k} of characteristic 0.

Recall that if A is a finite dimensional algebra over an algebraically closed field, then the *quiver* of A is the directed graph with vertices the isomorphism classes of simple A -modules and edges as follows. If S_1 and S_2 are simple A -modules, the number of directed edges from the isomorphism class of S_1 to the isomorphism class of S_2 is $\dim \text{Ext}_A^1(S_1, S_2)$. See [ARS97, Ben98, ASS06] for details.

Corollary 3.8. *The quiver of $\mathbb{C}\mathfrak{T}_n$ is acyclic for all $n \geq 1$.*

Proof. Proposition 3.5 implies that the only arrow exiting $L(1^r)$ is the arrow $L((1^r)) \rightarrow L((1^{r+1}))$ for $1 \leq r \leq n-1$ and that $L((1^n))$ is a sink. Proposition 3.6 implies that all other arrows go from a module with apex e_k to a module with apex e_j with $j < k$. It follows immediately that the quiver of $\mathbb{C}\mathfrak{T}_n$ is acyclic. \square

It is an open question to compute the quiver of $\mathbb{C}\mathfrak{T}_n$ for $n \geq 5$. The quiver for $1 \leq n \leq 4$ can be found in [Put98].

APPENDIX A. TILTING MODULES AND THE RINGEL DUAL FOR THE FULL TRANSFORMATION MONOID
BY VOLODYMYR MAZORCHUK AND BENJAMIN STEINBERG

A.1. Quasi-hereditary algebras and Ringel dual. Let \mathbb{k} be a field, A a finite dimensional \mathbb{k} -algebra and $\{L(\lambda) \mid \lambda \in \Lambda\}$ a fixed set of representatives of isomorphism classes of simple A -modules, where Λ is a fixed finite index set. For $\lambda \in \Lambda$, we fix an indecomposable projective cover $P(\lambda)$ of $L(\lambda)$ and an indecomposable injective envelope $I(\lambda)$ of $L(\lambda)$.

Fix a partial order $<$ on Λ and let \leq be the union of $<$ with the equality relation. Let $\Delta(\lambda)$, where $\lambda \in \Lambda$, denote the quotient of $P(\lambda)$ by the submodule generated by the images of all possible homomorphisms $P(\mu) \rightarrow P(\lambda)$, where $\mu \not\leq \lambda$. The modules $\Delta(\lambda)$, where $\lambda \in \Lambda$, are called *standard modules*. Let $\nabla(\lambda)$, where $\lambda \in \Lambda$, denote the submodule of $I(\lambda)$ defined as the intersection of the kernels of all possible homomorphisms $I(\lambda) \rightarrow I(\mu)$, where $\mu \not\leq \lambda$. The modules $\nabla(\lambda)$, where $\lambda \in \Lambda$, are called *costandard modules*.

The pair $(A, <)$ is called a *quasi-hereditary algebra*, see [Sco87, CPS88, DR89], provided that

- the endomorphism algebra of each $\Delta(\lambda)$ is a division algebra;
- each $P(\lambda)$ has a *standard filtration*, that is, a filtration whose subquotients are isomorphic to standard modules.

Equivalently, $(A, <)$ is quasi-hereditary provided that

- the endomorphism algebra of each $\nabla(\lambda)$ is a division algebra;
- each $I(\lambda)$ has a *costandard filtration*, that is, a filtration whose subquotients are isomorphic to costandard modules.

We refer the reader to the appendices in [DK94, Don98] and to [KK99] for more details.

Following [Rin91], an A -module T is called a *tilting module* provided that it has both a standard filtration and a costandard filtration. If $(A, <)$ is quasi-hereditary, then, for any $\lambda \in \Lambda$, there is a unique (up to isomorphism) indecomposable tilting module $T(\lambda)$ which contains $\Delta(\lambda)$ and such that the cokernel of the corresponding inclusion $\Delta(\lambda) \hookrightarrow T(\lambda)$ has a standard filtration. Moreover, every tilting module is isomorphic to a direct sum of (copies of) these $T(\lambda)$. The module

$$T = \bigoplus_{\lambda \in \Lambda} T(\lambda)$$

is called the *characteristic tilting module* for A . The algebra $\text{End}_A(T)^{\text{op}}$ is quasi-hereditary and is called the *Ringel dual* of A . We refer to [Rin91, KK99] for more details.

A.2. Representation theory of the full transformation monoid.

A.2.1. *The full transformation monoid.* We fix a field \mathbb{k} of characteristic 0 and $n \geq 1$. We will continue to use the notation of the previous sections. Simple $\mathbb{k}\mathfrak{T}_n$ -modules are parameterized by partitions $\lambda = (\lambda_1, \dots, \lambda_s)$ of r where $1 \leq r \leq n$. We denote by Λ the set of all such partitions. We write $|\lambda| = \lambda_1 + \dots + \lambda_s$. A special case of Putcha's results [Put98] shows that $\mathbb{k}\mathfrak{T}_n$ is quasi-hereditary with respect to the partial order that puts $\lambda < \rho$ if $|\lambda| > |\rho|$ for $\lambda, \rho \in \Lambda$. For each $\lambda \in \Lambda$, we fix a corresponding simple \mathfrak{T}_n -module $L(\lambda)$. For each $\lambda \in \Lambda$, we denote by $S(\lambda)$ the Specht $\mathbb{k}\mathfrak{S}_{|\lambda|}$ -module corresponding to λ .

A.2.2. *Standard and costandard $\mathbb{k}\mathfrak{T}_n$ -modules.* The following descriptions of the standard and costandard $\mathbb{k}\mathfrak{T}_n$ -modules (with respect to this quasi-hereditary structure) are well known, see for example [Put98]. We continue to use the idempotents e_1, \dots, e_n from earlier. The standard modules then turn out to be the induced modules and the costandard modules are the co-induced modules. More precisely, for a partition λ of r , we have that

$$\begin{aligned}\Delta(\lambda) &\cong \text{Ind}_{G_{e_r}}(S(\lambda)) \\ \nabla(\lambda) &\cong \text{Coind}_{G_{e_r}}(S(\lambda)).\end{aligned}$$

Note that, if λ is a partition of n , then we have $\Delta(\lambda) = L(\lambda) = \nabla(\lambda)$.

A.2.3. *(Co)standard versus simple $\mathbb{k}\mathfrak{T}_n$ -modules.* We have the *natural* $\mathbb{k}\mathfrak{T}_n$ -module $N = \mathbb{k}^n$ in which the module structure is defined by putting $fv_i = v_{f(i)}$, for $f \in \mathfrak{T}_n$, where v_1, \dots, v_n is the standard basis for \mathbb{k}^n . Let $\text{Aug}(N)$ be the *augmentation submodule* of N , it consists of all vectors whose coordinates sum to zero.

Theorem A.1. *Let λ be a partition of r with $1 \leq r \leq n$.*

- (i) *If $\lambda \neq (1^r)$, then $\Delta(\lambda) = L(\lambda)$.*
- (ii) *If $\lambda = (1^r)$, then $\Delta(\lambda) = P(\lambda) \cong \Lambda^r(N)$ and $L(\lambda) = \Lambda^{r-1}(\text{Aug}(N))$.*
- (iii) *For each $1 \leq r \leq n-1$, there is a short exact sequence*

$$0 \longrightarrow L((1^{r+1})) \longrightarrow P((1^r)) \longrightarrow L((1^r)) \longrightarrow 0. \quad (\text{A.1})$$

- (iv) *We have $P((1^n)) = L((1^n))$ is the one-dimensional sign representation of \mathfrak{S}_n , extended to \mathfrak{T}_n by sending all singular mappings to zero.*

Proof. Other than the claim $\Delta((1^r)) = P((1^r))$, the theorem is just a restatement of Theorem 2.10, Theorem 3.1 and Theorem 3.2. Corollary 3.3 implies that $P((1^r))$ has no composition factor $L(\lambda)$ with $\lambda \not\leq (1^r)$ and hence $P((1^r)) = \Delta((1^r))$ (alternatively, one can easily find a direct isomorphism between $\text{Ind}_{G_{e_r}}(S((1^r)))$ and $\Lambda^r(N)$). \square

A.3. Tilting modules for $\mathbb{k}\mathfrak{T}_n$ and the Ringel dual.

A.3.1. Multiplicities of $L((1^r))$ in injective modules. Our goal is to show that the indecomposable tilting modules for $\mathbb{k}\mathfrak{T}_n$ with respect to the quasi-hereditary structure mentioned above are precisely the injective indecomposable modules $I(\lambda)$, where $\lambda \neq (1)$, together with the simple projective module $L((1^n))$. We begin by studying multiplicities of $L((1^r))$ in injective modules. We shall write $[V : L]$ for the multiplicity of a simple module L as a composition factor of a $\mathbb{k}\mathfrak{T}_n$ -module V .

Proposition A.2. *Let $1 \leq r \leq n$ and let λ be a partition of r . Then we have*

$$[I(\lambda) : L((1^r))] = \begin{cases} 1, & \lambda = (1^r); \\ 1, & \lambda = (1^{r+1}), \ 1 \leq r \leq n-1; \\ 0, & \text{else.} \end{cases}$$

Proof. As \mathbb{k} is a splitting field for \mathfrak{T}_n , we have that

$$[I(\lambda) : L((1^r))] = \dim \operatorname{Hom}(P((1^r)), I(\lambda)) = [P((1^r)) : L(\lambda)].$$

Therefore, the claim follows directly from Theorem A.1(iii),(iv). \square

A.3.2. Directed injective tilting modules. From Proposition A.2, we easily deduce that a vast majority of the injective indecomposable modules are tilting modules.

Corollary A.3. *Let $1 \leq r \leq n$ and λ be a partition of r different from (1^r) . Then $T(\lambda) = I(\lambda)$.*

Proof. Any injective indecomposable module for a quasi-hereditary algebra always has a filtration by costandard modules. By Proposition A.2, each composition factor of $I(\lambda)$ is of the form $L(\nu)$ with ν not of the form (1^s) . But then $\Delta(\nu) = L(\nu)$ by Theorem A.1 and so a composition series for $I(\lambda)$ is a filtration by standard modules. Thus $I(\lambda)$ is an indecomposable tilting module with socle $L(\lambda) = \Delta(\lambda)$. We conclude that $I(\lambda) = T(\lambda)$. \square

A.3.3. Indecomposable tilting $\mathbb{k}\mathfrak{T}_n$ -modules. We can now describe all indecomposable tilting $\mathbb{k}\mathfrak{T}_n$ -modules.

Theorem A.4. *Let \mathbb{k} be a field of characteristic 0 and λ a partition of r , where $1 \leq r \leq n$. Then we have*

$$T(\lambda) = \begin{cases} L((1^n)), & \text{if } \lambda = (1^n); \\ I(1^{r+1}), & \text{if } \lambda = (1^r), \ 1 \leq r \leq n-1; \\ I(\lambda), & \text{else.} \end{cases}$$

Proof. Since $P((1^n)) = L((1^n)) = \Delta((1^n)) = \nabla((1^n))$, we have $T((1^n)) = L((1^n))$. Hence, by Corollary A.3, it remains to show that $T((1^r)) = I((1^{r+1}))$, for every $1 \leq r \leq n-1$.

The module $I((1^{r+1}))$ has simple socle $L((1^{r+1}))$, which is a submodule of $\Delta((1^r)) = P((1^r))$ by (A.1). Hence, by injectivity of $I((1^{r+1}))$,

the inclusion $L((1^{r+1})) \hookrightarrow I((1^{r+1}))$ extends to a non-zero homomorphism $\varphi: \Delta((1^r)) \rightarrow I((1^{r+1}))$, which must be injective because $L((1^{r+1})) = \text{rad}(\Delta((1^r)))$ is the unique maximal submodule of $\Delta((1^r))$ by (A.1). Since $\Delta((1^r)) = P((1^r))$ has exactly two composition factors by Theorem A.1(iii), namely $L((1^r))$ and $L((1^{r+1}))$, we deduce from Proposition A.2 that the cokernel $V = I((1^{r+1}))/\Delta((1^r))$ has no composition factor of the form $L((1^s))$, where $1 \leq s \leq n$. Thus, by Theorem A.1, every simple subquotient of V is a standard module and hence V has a standard filtration. As $I((1^{r+1}))$ has a costandard filtration (being injective), we obtain that $I((1^{r+1}))$ is a tilting module. Moreover, from the definitions we also have that $I((1^{r+1})) \cong T((1^r))$. \square

A.3.4. The Ringel dual. Let I_1 be the ideal of \mathfrak{T}_n consisting of the constant mappings. We show that the Ringel dual of $\mathbb{k}\mathfrak{T}_n$ with respect to the quasi-hereditary structure we have been considering is Morita equivalent to a one-point extension of $\mathbb{k}\mathfrak{T}_n/\mathbb{k}I_1$. Here we use that the Ringel dual is Morita equivalent to $\text{End}_{\mathbb{k}\mathfrak{T}_n}(T')^{op}$ for any tilting module T' that contains each $T(\lambda)$ as a direct summand.

Let D be the standard duality between right and left $\mathbb{k}\mathfrak{T}_n$ -modules, so $D(V) = \text{Hom}_{\mathbb{k}}(V, \mathbb{k})$ for a right/left $\mathbb{k}\mathfrak{T}_n$ -module. Note that D sends projective/injective modules to injective/projective modules and simple modules to simple modules. Let $P'(\lambda)$ denote the right projective cover of the simple module $L'(\lambda) = D(L(\lambda))$. Then as a right module

$$\mathbb{k}\mathfrak{T}_n = \bigoplus_{\lambda \in \Lambda} P'(\lambda)^{\dim L'(\lambda)}$$

and, in particular, $P'((1))$ appears with multiplicity one in this decomposition. Note that the constant mapping e_1 is a primitive idempotent with $P'((1)) = e_1 \mathbb{k}\mathfrak{T}_n = L'((1))$. To ease notation, we put $e = e_1$. Then

$$(1 - e) \mathbb{k}T_n = \bigoplus_{\lambda \in \Lambda \setminus \{(1)\}} P'(\lambda)^{\dim L'(\lambda)} \quad (\text{A.2})$$

and if $V = L'((1^n)) \oplus (1 - e) \mathbb{k}T_n$, then

$$T' = D(V) = L((1^n)) \oplus \bigoplus_{\lambda \in \Lambda \setminus \{(1)\}} I(\lambda)^{\dim L(\lambda)}$$

is a tilting module containing each indecomposable tilting module as a direct summand by Theorem A.4. Thus $A = \text{End}_{\mathbb{k}\mathfrak{T}_n}(T')^{op} \cong \text{End}_{\mathbb{k}\mathfrak{T}_n^{op}}(V)$ is Morita equivalent to the Ringel dual of $\mathbb{k}\mathfrak{T}_n$.

Clearly, we have

$$\begin{aligned} A &\cong \begin{bmatrix} \text{End}_{\mathbb{k}\mathfrak{T}_n^{op}}(L'((1^n))) & \text{Hom}_{\mathbb{k}\mathfrak{T}_n^{op}}((1 - e) \mathbb{k}\mathfrak{T}_n, L'((1^n))) \\ \text{Hom}_{\mathbb{k}\mathfrak{T}_n^{op}}(L'((1^n)), (1 - e) \mathbb{k}\mathfrak{T}_n) & \text{End}_{\mathbb{k}\mathfrak{T}_n^{op}}((1 - e) \mathbb{k}\mathfrak{T}_n) \end{bmatrix} \\ &\cong \begin{bmatrix} \mathbb{k} & L'((1^n))(1 - e) \\ 0 & (1 - e) \mathbb{k}\mathfrak{T}_n(1 - e) \end{bmatrix} \end{aligned}$$

because $\text{Hom}_{\mathbb{k}\mathfrak{T}_n^{\text{op}}}(L'((1^n)), (1-e)\mathbb{k}\mathfrak{T}_n) = 0$ by the dual of Proposition A.2, as $[(1-e)\mathbb{k}\mathfrak{T}_n : L'((1^n))] = 1$ and the occurrence is as the simple top of $P'((1^n))$, not in the socle.

Now if $f, g \in \mathfrak{T}_n$, then $(1-e)f(1-e) = f - fe$ and $(f - fe)(g - ge) = fg - fge$. Also, note that $f - fe = 0$ for any constant mapping f . Thus there is an isomorphism from $\mathbb{k}\mathfrak{T}_n/\mathbb{k}I_1$ to $(1-e)\mathfrak{T}_n(1-e)$ sending the coset of $f \in \mathfrak{T}_n \setminus I_1$ to $f - fe$. The coset of f acts on the right of $L'((1^n))(1-e)$ by multiplication by $\text{sgn}(f)$ where we extend sgn to \mathfrak{T}_n by sending non-permutations to 0. In conclusion, we have proved the following theorem.

Theorem A.5. *The Ringel dual of $\mathbb{k}\mathfrak{T}_n$ is Morita equivalent to the one-point extension of $\mathbb{k}\mathfrak{T}_n/\mathbb{k}I_1$*

$$\begin{bmatrix} \mathbb{k} & \mathbb{k} \\ 0 & \mathbb{k}\mathfrak{T}_n/\mathbb{k}I_1 \end{bmatrix}$$

where I_1 is the ideal of constant mappings and where \mathbb{k} is made a right $\mathbb{k}\mathfrak{T}_n/\mathbb{k}I_1$ -module via the extension of the sign representation of \mathfrak{S}_n to \mathfrak{T}_n that vanishes on $\mathfrak{T}_n \setminus \mathfrak{S}_n$ (and, in particular, on I_1).

REFERENCES

- [ARS97] M. Auslander, I. Reiten, and S. O. Smalø. *Representation theory of Artin algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. Corrected reprint of the 1995 original.
- [ASS06] I. Assem, D. Simson, and A. Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*, volume 65 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006. Techniques of representation theory.
- [Ben98] D. J. Benson. *Representations and cohomology. I*, volume 30 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1998. Basic representation theory of finite groups and associative algebras.
- [CP61] A. H. Clifford and G. B. Preston. *The algebraic theory of semigroups. Vol. I*. Mathematical Surveys, No. 7. American Mathematical Society, Providence, R.I., 1961.
- [CPS88] E. Cline, B. Parshall, and L. Scott. Finite-dimensional algebras and highest weight categories. *J. Reine Angew. Math.*, 391:85–99, 1988.
- [CSST10] T. Ceccherini-Silberstein, F. Scarabotti, and F. Tolli. *Representation theory of the symmetric groups*, volume 121 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010. The Okounkov-Vershik approach, character formulas, and partition algebras.
- [CTVEZ03] J. F. Carlson, L. Townsley, L. Valeri-Elizondo, and Mucheng Zhang. *Cohomology rings of finite groups*, volume 3 of *Algebras and Applications*. Kluwer Academic Publishers, Dordrecht, 2003. With an appendix: Calculations of cohomology rings of groups of order dividing 64 by Carlson, Valeri-Elizondo and Zhang.
- [Dia88] P. Diaconis. *Group representations in probability and statistics*. Institute of Mathematical Statistics Lecture Notes—Monograph Series, 11. Institute of Mathematical Statistics, Hayward, CA, 1988.
- [Dia89] P. Diaconis. A generalization of spectral analysis with application to ranked data. *Ann. Statist.*, 17(3):949–979, 1989.

- [DK94] Y. A. Drozd and Vladimir V. Kirichenko. *Finite-dimensional algebras*. Springer-Verlag, Berlin, 1994. Translated from the 1980 Russian original and with an appendix by Vlastimil Dlab.
- [Don98] S. Donkin. *The q -Schur algebra*, volume 253 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.
- [DR89] V. Dlab and C. M. Ringel. Quasi-hereditary algebras. *Illinois J. Math.*, 33(2):280–291, 1989.
- [Eil76] S. Eilenberg. *Automata, languages, and machines. Vol. B*. Academic Press, New York, 1976. With two chapters (“Depth decomposition theorem” and “Complexity of semigroups and morphisms”) by Bret Tilson, *Pure and Applied Mathematics*, Vol. 59.
- [FH91] W. Fulton and J. Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.
- [GM09] O. Ganyushkin and V. Mazorchuk. *Classical finite transformation semigroups, an introduction*. Number 9 in *Algebra and Applications*. Springer, 2009.
- [GMS09] O. Ganyushkin, V. Mazorchuk, and B. Steinberg. On the irreducible representations of a finite semigroup. *Proc. Amer. Math. Soc.*, 137(11):3585–3592, 2009.
- [Gre51] J. A. Green. On the structure of semigroups. *Ann. of Math. (2)*, 54:163–172, 1951.
- [Gre80] J. A. Green. *Polynomial representations of GL_n* , volume 830 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1980.
- [Hig92] P. M. Higgins. *Techniques of semigroup theory*. Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1992. With a foreword by G. B. Preston.
- [HZ57] E. Hewitt and H. S. Zuckerman. The irreducible representations of a semi-group related to the symmetric group. *Illinois J. Math.*, 1:188–213, 1957.
- [JK81] G. James and A. Kerber. *The representation theory of the symmetric group*, volume 16 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1981. With a foreword by P. M. Cohn, With an introduction by Gilbert de B. Robinson.
- [KK99] M. Klucznik and S. König. Characteristic tilting modules over quasi-hereditary algebras. Preprint 99004, SFB 343, Bielefeld University, Bielefeld, Germany, 1999.
- [KRT68] K. Krohn, J. Rhodes, and B. Tilson. *Algebraic theory of machines, languages, and semigroups*. Edited by Michael A. Arbib. With a major contribution by Kenneth Krohn and John L. Rhodes. Academic Press, New York, 1968. Chapters 1, 5–9.
- [Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.
- [MS11] S. Margolis and B. Steinberg. The quiver of an algebra associated to the Mantaci-Reutenauer descent algebra and the homology of regular semigroups. *Algebr. Represent. Theory*, 14(1):131–159, 2011.
- [Mun57] W. D. Munn. Matrix representations of semigroups. *Proc. Cambridge Philos. Soc.*, 53:5–12, 1957.
- [Nic71] W. R. Nico. Homological dimension in semigroup algebras. *J. Algebra*, 18:404–413, 1971.
- [Nic72] W. R. Nico. An improved upper bound for global dimension of semigroup algebras. *Proc. Amer. Math. Soc.*, 35:34–36, 1972.

- [Pon56] I. S. Ponizovskii. On matrix representations of associative systems. *Mat. Sb. N.S.*, 38(80):241–260, 1956.
- [Pon87] I. S. Ponizovskii. Some examples of semigroup algebras of finite representation type. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 160(Anal. Teor. Chisel i Teor. Funktsii. 8):229–238, 302, 1987.
- [Put96] M. S. Putcha. Complex representations of finite monoids. *Proc. London Math. Soc. (3)*, 73(3):623–641, 1996.
- [Put98] M. S. Putcha. Complex representations of finite monoids. II. Highest weight categories and quivers. *J. Algebra*, 205(1):53–76, 1998.
- [Rin91] C. M. Ringel. The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. *Math. Z.*, 208(2):209–223, 1991.
- [Rin00] C. M. Ringel. The representation type of the full transformation semigroup T_4 . *Semigroup Forum*, 61(3):429–434, 2000.
- [RS09] J. Rhodes and B. Steinberg. *The q -theory of finite semigroups*. Springer Monographs in Mathematics. Springer, New York, 2009.
- [Sag01] B. E. Sagan. *The symmetric group*, volume 203 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001. Representations, combinatorial algorithms, and symmetric functions.
- [Sco87] L. L. Scott. Simulating algebraic geometry with algebra. I. The algebraic theory of derived categories. In *The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986)*, volume 47 of *Proc. Sympos. Pure Math.*, pages 271–281. Amer. Math. Soc., Providence, RI, 1987.
- [Ste] B. Steinberg. *The representation theory of finite monoids*. Springer. To appear.
- [Til76] B. Tilson. *Depth decomposition theorem*, chapter XI, pages 287–312. In Eilenberg [Eil76], 1976.

BS: DEPARTMENT OF MATHEMATICS, CITY COLLEGE OF NEW YORK, CONVENT AVENUE AT 138TH STREET, NEW YORK, NEW YORK 10031, USA

E-mail address: `bsteinberg@ccny.cuny.edu`

VM: DEPARTMENT OF MATHEMATICS, UPPSALA UNIVERSITY, BOX 480, SE-75106,, UPPSALA, SWEDEN

E-mail address: `mazor@math.uu.se`